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CONTINUOUS THEORY OF DISLOCATIONS AND DISCLINATIONS IN A TWO-DIMENSIONAL MEDIUM*

YU.Z. POVSTENKO

A system of equations describing mobile defects in a two-dimensional Cosser at continuum, i.e. in a medium whose motion is determined by the displacement field and rotation field independent of it, is obtained.

The basic equations of the static theory /1—5/ and dynamic continuous theory /6—12/ of defects (dislocations and disclinations) are known for a three-dimensional medium, obtained by a variety of methods. A dislocation model of the misalignment surfaces used in describing the Martensitic transformations /2, 13/ is proposed. The dislocation representations were used in /14—16/ to describe the grain boundaries, and the difference dislocations within the boundaries of separation were studied in /17, 18/. The dislocation structure of internal boundaries of separation was described in /19, 20/ using the differential geometry characteristics (torsion and curvature tensors, non-holonomic object) of three-dimensional media. Surface dislocations and disclinations of the separate Volterra distortions-type were studied in /21/, with liquid crystals and various biological objects indicated as the suitable areas of application of these concepts.

1. Surface del operator. A surface imbedded in a three-dimensional Euclidean space is described by the equations $x^i = x^i(y^1, y^2)$ where y^α are curvilinear coordinates on the surface. Henceforth, the Latin indices will take the values of 1, 2, 3, and the Greek indices values of 1, 2. Regarding the radius vector $\mathbf{r}$ of a point on the surface as a function of the coordinates y^α, we introduce the local tangential basis vectors $\mathbf{a}_\alpha = \partial\mathbf{r}/\partial y^\alpha$ and the normal vector $\mathbf{n} = {}^1/_2\, \varepsilon^{\alpha\beta}\mathbf{a}_\alpha \times \mathbf{a}_\beta$ where $\varepsilon^{\alpha\beta}$ are the components of the Levi-Civita surface vector $\boldsymbol{\varepsilon}_\Sigma = \varepsilon^{\alpha\beta}\mathbf{a}_\alpha\mathbf{a}_\beta$.

The surface del operator /22/

$$\nabla_\Sigma = \mathbf{a}^\alpha \partial/\partial y^\alpha$$

enables us to define, for the tensor $\mathbf{T}_\Sigma$ defined on the surface, the operations of surface grad, div and curl

$$\mathrm{grad}_\Sigma \mathbf{T}_\Sigma \equiv \nabla_\Sigma \mathbf{T}_\Sigma, \quad \mathrm{div}_\Sigma \mathbf{T}_\Sigma \equiv \nabla_\Sigma \cdot \mathbf{T}_\Sigma, \quad \mathrm{rot}_\Sigma \mathbf{T}_\Sigma \equiv \nabla_\Sigma \times \mathbf{T}_\Sigma$$

The rules of action of the surface del operator on the products of the quantities are identical to those of the three-dimensional del operator $\nabla = \mathbf{a}^k \partial/\partial x^k$ (see e.g. /23/. Essential differences due to the surface curvature appear on the second application of the two-dimensional del operator. For example, the following relations hold:

$$\nabla_\Sigma \times (\nabla_\Sigma \mathbf{T}_\Sigma) = \boldsymbol{\varepsilon}_\Sigma \cdot \mathbf{b} \cdot \nabla_\Sigma \mathbf{T}_\Sigma \tag{1.1}$$

$$\nabla_\Sigma \cdot (\nabla_\Sigma \times \mathbf{T}_\Sigma) = -2H\mathbf{n} \cdot (\nabla_\Sigma \times \mathbf{T}_\Sigma) + \nabla_\Sigma \cdot (\boldsymbol{\varepsilon}_\Sigma \cdot \mathbf{b} \cdot \mathbf{T}_\Sigma) \tag{1.2}$$

while in the three-dimensional case we have

$$\nabla \times (\nabla \mathbf{T}) = 0, \quad \nabla \cdot (\nabla \times \mathbf{T}) = 0 \tag{1.3}$$

Here $\mathbf{b} = b_{\alpha\beta}\mathbf{a}^\alpha\mathbf{a}^\beta$ is the tensor of the second quadratic form of the surface and $H = {}^1/_2 b_\alpha{}^\alpha$. is the mean surface curvature.

2. Defects in the three-dimensional Cosserat continuum. To order to facilitate the presentation of the corresponding results for the two-dimensional Cosserat continuum, we shall give the basic equations for the three-dimensional medium (e.g. /24—26/).

The non-symmetric total deformation γ and flexure-torsion $\varkappa$ tensors are expressed in terms of the displacement $\mathbf{u}$ and rotation $\boldsymbol{\omega}$ vector thus

$$\boldsymbol{\gamma} = \nabla\mathbf{u} + \mathbf{g} \times \boldsymbol{\omega}, \quad \boldsymbol{\varkappa} = \nabla\boldsymbol{\omega}$$

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and satisfy the conditions of compatibility following from the first formula of (1.3)

$$\nabla\times\gamma-\varkappa^*+(\mathrm{tr}\,\varkappa)\,\mathbf{g}=0\quad(\mathrm{tr}\,\varkappa\equiv\mathbf{g}:\varkappa),\quad \nabla\times\varkappa=0$$

The asterisk denotes transposition, and $\mathbf{g}$ is a metric tensor.

Let us write the quantities γ and $\varkappa$ in the form of a sum of the elastic and plastic components denoted, respectively, by the indices e and p, and introduce the dislocation α and disclination θ density tensors

$$\alpha=-\nabla\times\gamma^p+\varkappa^{p*}-(\mathrm{tr}\,\varkappa^p)\,\mathbf{g},\quad \theta=-\nabla\times\varkappa^p \tag{2.1}$$

satisfying, on the basis of the second relation of (1.3), the conditions (ε is a three-dimensiona Levi-Civita tensor)

$$\nabla\cdot\alpha-\varepsilon:\theta=0,\ \nabla\cdot\theta=0 \tag{2.2}$$
$$(\nabla_k\alpha^{km}-\varepsilon^{mij}\theta_{ij}=0,\ \nabla_k\theta^{km}=0)$$

The above conditions imply that the disclinations do not terminate within the body and the dislocation may terminate on the disclinations whose density is an asymmetric tensor /26/.

In the linear theory we have (a dot denotes time differentiation)

$$\gamma^{\cdot}=\nabla\mathbf{v}+\mathbf{g}\times\mathbf{w},\quad \varkappa^{\cdot}=\nabla\mathbf{w}\ (\mathbf{v}=\mathbf{u}^{\cdot},\ \mathbf{w}=\boldsymbol{\omega}^{\cdot}) \tag{2.3}$$

The tensors of the dislocation flux $\mathbf{J}$ and disclination flux $\mathbf{S}$ are introduced as follows /11/:

$$\mathbf{J}=\gamma^{\cdot p}-\nabla\mathbf{v}^p-\mathbf{g}\times\mathbf{w}^p,\quad \mathbf{S}=\varkappa^{\cdot p}-\nabla\mathbf{w}^p \tag{2.4}$$

and the formulas (2.1),(2.3), (2.4) yield the kinematic equations

$$\alpha^{\cdot}=-\nabla\times\mathbf{J}+\mathbf{S}^*-(\mathrm{tr}\,\mathbf{S})\,\mathbf{g},\quad \theta^{\cdot}=-\nabla\times\mathbf{S} \tag{2.5}$$
$$(\alpha^{\cdot km}=-\varepsilon^{kij}\nabla_iJ_{j\cdot}^{\ m}+S^{mk}-S_{i\cdot}^{\ i}g^{km},\quad \theta^{\cdot km}=-\varepsilon^{kij}\nabla_iS_{j\cdot}^{\ m})$$

3. Defects in the two-dimensional Cosserat continuum.

The non-symmetric tensors of total deformation γ_Σ and flexure-torsion $\varkappa_\Sigma$ are expressed in terms of the displacement $\mathbf{u}_\Sigma$ and rotation ω_Σ vectors as follows:

$$\gamma_\Sigma=\nabla_\Sigma\mathbf{u}_\Sigma+\mathbf{a}\times\omega_\Sigma,\quad \varkappa_\Sigma=\nabla_\Sigma\omega_\Sigma \tag{3.1}$$

and satisfy, by virtue of (1.1), the conditions of compatibility ($\mathbf{a}$ is the metric tensor on the surface)

$$\nabla_\Sigma\times\gamma_\Sigma-\varepsilon_\Sigma\cdot\mathbf{b}\cdot\gamma_\Sigma+\mathbf{nn}\,\mathrm{tr}_\Sigma\,\varkappa_\Sigma-\mathbf{nn}\cdot\varkappa_\Sigma^*=0 \tag{3.2}$$
$$\nabla_\Sigma\times\varkappa_\Sigma-\varepsilon_\Sigma\cdot\mathbf{b}\cdot\varkappa_\Sigma=0\quad(\mathrm{tr}_\Sigma\,\varkappa_\Sigma\equiv\mathbf{a}:\varkappa_\Sigma)$$

The surface dislocation α_Σ and disclination θ_Σ density tensors

$$\alpha_\Sigma=-\nabla_\Sigma\times\gamma_\Sigma^p+\varepsilon_\Sigma\cdot\mathbf{b}\cdot\gamma_\Sigma^p-\mathbf{nn}\,\mathrm{tr}_\Sigma\varkappa_\Sigma^p+\mathbf{nn}\cdot\varkappa_\Sigma^{p*} \tag{3.3}$$
$$\theta_\Sigma=-\nabla_\Sigma\times\varkappa_\Sigma^p+\varepsilon_\Sigma\cdot\mathbf{b}\cdot\varkappa_\Sigma^p$$

satisfy, by virtue of (1.2), the conditions

$$\nabla_\Sigma\cdot\alpha_\Sigma+2H\mathbf{n}\cdot\alpha_\Sigma=0,\quad \nabla_\Sigma\cdot\theta_\Sigma+2H\mathbf{n}\cdot\theta_\Sigma=0 \tag{3.4}$$

Let us define the surface dislocation flux $\mathbf{J}_\Sigma$ and disclination flux $\mathbf{S}_\Sigma$ tensors

$$\mathbf{J}_\Sigma=\gamma_\Sigma^{\cdot p}-\nabla_\Sigma\mathbf{v}_\Sigma^p-\mathbf{a}\times\mathbf{w}_\Sigma^p,\quad \mathbf{S}_\Sigma=\varkappa_\Sigma^{\cdot p}-\nabla_\Sigma\mathbf{w}_\Sigma^p \tag{3.5}$$

Then we obtain for $\alpha_\Sigma^{\cdot}$ and $\theta_\Sigma^{\cdot}$ the two-dimensional analogues of the equations (2.5)

$$\alpha_\Sigma^{\cdot}=-\nabla_\Sigma\times\mathbf{J}_\Sigma+\varepsilon_\Sigma\cdot\mathbf{b}\cdot\mathbf{J}_\Sigma-\mathbf{nn}\,\mathrm{tr}_\Sigma\,\mathbf{S}_\Sigma+\mathbf{nn}\cdot\mathbf{S}_\Sigma^* \tag{3.6}$$
$$\theta_\Sigma^{\cdot}=-\nabla_\Sigma\times\mathbf{S}_\Sigma+\varepsilon_\Sigma\cdot\mathbf{b}\cdot\mathbf{S}_\Sigma$$

or, in terms of the components,

$$\alpha^{\cdot(n)\gamma}=-\varepsilon^{\alpha\beta}(\nabla_\alpha J_{\beta\cdot}^{\ \gamma}-b_{\alpha\cdot}^{\ \gamma}J_{\beta(n)})+S^{\gamma(n)} \tag{3.7}$$
$$\alpha^{\cdot(n)(n)}=-\varepsilon^{\alpha\beta}(\nabla_\alpha J_{\beta(n)}+b_{\alpha\gamma}J_{\beta\cdot}^{\ \gamma})-S_{\alpha\cdot}^{\ \alpha}$$
$$\theta^{\cdot(n)\gamma}=-\varepsilon^{\alpha\beta}(\nabla_\alpha S_{\beta\cdot}^{\ \gamma}-b_{\alpha\cdot}^{\ \gamma}S_{\beta(n)})$$
$$\theta^{\cdot(n)(n)}=-\varepsilon^{\alpha\beta}(\nabla_\alpha S_{\beta(n)}+b_{\alpha\gamma}S_{\beta\cdot}^{\ \gamma})$$

We note that the first index accompanying the tensors $\gamma_\Sigma,\varkappa_\Sigma,\mathbf{J}_\Sigma$ and $\mathbf{S}_\Sigma$ is the surface index, while the second index is, in general, spatial, and the first index accompanying the tensors α_Σ and θ_Σ always refers to the normal to the surface, while the second one is spatial.

Taking the structure of the tensors α_Σ and θ_Σ into account, we conclude that Eqs.(3.4) will be satisfied identically: for a two-dimensional continuum the dislocation and disclination lines are directed along the normal to the surface and are, naturally, not terminated within the body.

4. Burgers and Frank surface dislocation and disclination vectors.

The Burgers vector $\mathbf{b}_\Sigma$ and Frank vector Ω_Σ are defined as follows:

$$[\mathbf{u}_\Sigma]=\oint_{C_\Sigma}d\mathbf{u}_\Sigma=\mathbf{b}_\Sigma+\Omega_\Sigma\times\mathbf{r}_{0\Sigma},\quad [\omega_\Sigma]=\oint_{C_\Sigma}d\omega_\Sigma=\Omega_\Sigma \tag{4.1}$$

where C_Σ is the Burgers contour lying on the surface and $\mathbf{r}_{0\Sigma}$ is the radius vector of the beginning and end of the reading on the contour C_Σ.

Since $d\mathbf{u}_\Sigma = d\mathbf{r}\cdot\nabla_\Sigma \mathbf{u}_\Sigma$, $d\boldsymbol{\omega}_\Sigma = d\mathbf{r}\cdot\nabla_\Sigma \boldsymbol{\omega}_\Sigma$, we obtain, using Eqs.(3.1) and Stokes' formula

$$\mathbf{b}_\Sigma = \int_\Sigma \mathbf{n}\cdot[\nabla_\Sigma \times \boldsymbol{\gamma}_\Sigma - (\nabla_\Sigma \times \boldsymbol{\varkappa}_\Sigma)\times \mathbf{r} - \mathbf{nn}\cdot\boldsymbol{\varkappa}_\Sigma^* + \mathbf{nn}\,\mathrm{tr}_\Sigma\,\boldsymbol{\varkappa}_\Sigma]\,d\Sigma, \quad \boldsymbol{\Omega}_\Sigma = \int_\Sigma \mathbf{n}\cdot(\nabla_\Sigma \times \boldsymbol{\varkappa}_\Sigma)\,d\Sigma \qquad (4.2)$$

Substituting (3.3) into (4.2) and remembering that $\mathbf{n}\cdot\mathbf{e}_\Sigma\cdot\mathbf{b}\cdot\boldsymbol{\gamma}_\Sigma = \mathbf{n}\cdot\mathbf{e}_\Sigma\cdot\mathbf{b}\cdot\boldsymbol{\varkappa}_\Sigma = 0$, we obtain the two-dimensional analogues of the corresponding three-dimensional formulas for the Burgers and Frank surface defect vectors

$$\mathbf{b}_\Sigma = \int_\Sigma \mathbf{n}\cdot(\boldsymbol{\alpha}_\Sigma - \boldsymbol{\theta}_\Sigma \times \mathbf{r})\,d\Sigma, \quad \boldsymbol{\Omega}_\Sigma = \int_\Sigma \mathbf{n}\cdot\boldsymbol{\theta}_\Sigma\,d\Sigma \qquad (4.3)$$

Following the terminology of /21/, we shall call the dislocations with Burgers vectors lying in the tangent plane (normal to the surface) and disclinations with Frank vectors normal to the surface (lying in the tangent plane), the internal (external) defects.

5. Connection with non-Riemannian geometry. Three-dimensional continuum.

We will determine, in the space of affine connectivity, for the tensor $\mathbf{T}$ with components $T^k_{\cdot m}(\xi^i, \tau)$, the tensor $D\mathbf{T}$ with components

$$(D\mathbf{T})^k_{\cdot m} = \nabla_i T^k_{\cdot m}\, d\xi^i$$

where the covariant derivative is calculated using the affine connectivity coefficients $\Gamma_{ij}^{\ \ k}$

$$\nabla_i T^k_{\cdot m} = \partial T^k_{\cdot m}/\partial\xi^i + \Gamma^k_{iq} T^q_{\cdot m} - \Gamma^q_{im} T^k_{\cdot q}$$

The quantities $\Gamma_{ij}^{\ \ k}$ are, generally speaking, non-symmetric with respect to the lower indices, and the antisymmetric part $\Gamma^k_{[ij]}$ defines the torsion tensor $\Gamma^k_{[ij]} = S_{ij}^{\ \ k}$.

The following relation holds for the tensor $\mathbf{V} = D'D\mathbf{T} - DD'\mathbf{T}$, ($R_{rsl}^{\ \ \ k}$ are the components of the curvature tensor)

$$V^k_{\cdot m} = (R_{rsp}^{\ \ \ k} T^p_{\cdot m} - R_{rsm}^{\ \ \ p} T^k_{\cdot p})\, d\xi'^r\, d\xi^s \qquad (5.1)$$

$$R_{rsp}^{\ \ \ k} = 2\,(\partial_{[r}\Gamma^k_{s]p} + \Gamma^k_{[r|q|}\Gamma^q_{s]p}) \qquad (5.2)$$

The curvature and torsion tensors satisfy the Bianchi-Padov identity /27/.

In the geometrical theory of defects the torsion tensor $S_{jq}^{\ \ k}$ is placed in correspondence with the dislocation density tensor α^{mk} /1. 2/, and the curvature tensor R_{pqrs} with the disclination density tensor θ^{mk} /5/

$$\alpha^{mk} = \varepsilon^{mpq} S_{pq}^{\ \ k}, \quad \theta^{mn} = \tfrac{1}{4}\varepsilon^{mpq}\varepsilon^{nrs} R_{pqrs}$$

and the Bianchi-Padov identity yields the Eqs.(5)

$$\nabla_k \alpha^{km} - \varepsilon^{mnq}\theta_{nq} = \varepsilon_{ijs}\alpha^{ij}\alpha^{km}, \quad \nabla_k \theta^{km} = \varepsilon_{ijs}\alpha^{ij}\theta^{km}$$

representing the non-linear generalization of Eqs.(2.2).

Following /12/ we introduce the tensor $D^\tau\mathbf{T}$ with components

$$(D^\tau\mathbf{T})^k_{\cdot m} = \nabla^\tau T^k_{\cdot m}\, d\tau$$

$$\nabla^\tau T^k_{\cdot m} = T^{\cdot k}_{\cdot m} + h_q^{\ k} T^q_{\cdot m} - h_m^{\ q} T^k_{\cdot q}$$

The components of the tensor $h_q^{\ k}$ can be regarded as components of the time derivatives of the local basis vectors in tangential space at the corresponding point. In order not to increase the notation employed, we shall write the tensor components in the Lagrangian form without the "roofs".

The tensor $\mathbf{W} = D^\tau D\mathbf{T} - DD^\tau\mathbf{T}$ has the following components:

$$W^k_{\cdot m} = (P_{sq}^{\ \ k} T^q_{\cdot m} - P_{sm}^{\ \ q} T^k_{\cdot q})\, d\tau\, d\xi^s \qquad (5.3)$$

where

$$P_{ik}^{\ \ m} = \Gamma^{\cdot m}_{ik} - \nabla_i h_k^{\ m} \qquad (5.4)$$

Thus we find, in the space of affine connectivity whose properties vary with time, in addition to the curvature and torsion tensors, two new characteristics, namely the tensors $h_k^{\ m}$ and $P_{sk}^{\ \ m}$.

Formulas (5.2) and (5.4) yield the evolutionary equations for the components of the torsion and curvature tensors

$$\dot{S}_{ik}^{\ \ m} = \nabla_{[i} h_{k]}^{\ \ m} + P_{[ik]}^{\ \ \ m} \qquad (5.5)$$

$$\dot{R}_{rsk}^{\ \ \ m} = 2\,[\nabla_{[r} P_{s]k}^{\ \ \ m} + \nabla_{[r}\nabla_{s]} h_k^{\ m} + S_{rs}^{\ \ i}(P_{ik}^{\ \ m} + \nabla_i h_k^{\ m})]$$

In the Euclidean space $S_{ik}^{\ \ m} = 0$, $R_{rsk}^{\ \ \ m} = 0$, $P_{ik}^{\ \ m} = 0$, $h_k^{\ m} = \nabla_k v^m$ where v^m are the velocity vector components, and from formula (5.4), it follows that $\Gamma^{\cdot m}_{ik} = \nabla_i \nabla_k v^m$.

In the general case the tensors $h_{k\cdot}^{m}$ and $P_{sk\cdot}^{m}$ are connected with the dislocation flux $J_{k\cdot}^{m}$ and the disclination flux $S_{k\cdot}^{m}$ as follows:

$$J_{k\cdot}^{m} = \nabla_k v^m - h_{k\cdot}^{m}, \quad S_{k\cdot}^{m} = -\tfrac{1}{2}\varepsilon^{mpq} P_{kpq} \tag{5.6}$$

The choice of sign in these formulas is a matter of choice. Unlike in /12/, in the present paper the sign is chosen so that the sign of J and S in the last formulas coincides with the sign of the corresponding quantities in (2.4).

The evolutionary Eqs.(5.5) yield the non-linear equations of the continuous theory of mobile defects /12/. Neglecting the non-linear terms in them yields the conditions (2.5).

6. Relation to non-Riemannian geometry. Two-dimensional continuum.

Let a tensor $\mathbf{T}_\Sigma$ with components $T^{\alpha}_{\cdot\beta}(\eta^\gamma, \tau)$ be given on the surface Σ with normal $\mathbf{n}$. The components of the tensor $D\mathbf{T}_\Sigma$ have the form

$$(D\mathbf{T}_\Sigma)^{\alpha}_{\cdot\beta} = \nabla_\gamma T^{\alpha}_{\cdot\beta}\, d\eta^\gamma, \quad (D\mathbf{T}_\Sigma)^{\alpha(n)} = T^{\alpha}_{\cdot\beta} b_{\gamma\cdot}^{\beta}\, d\eta^\gamma$$

$$(D\mathbf{T}_\Sigma)_{(n)\beta} = T^{\alpha}_{\cdot\beta} b_{\gamma\alpha}\, d\eta^\gamma$$

where the covariant derivative

$$\nabla_\gamma T^{\alpha}_{\cdot\beta} = \partial T^{\alpha}_{\cdot\beta}/\partial\eta^\gamma + G^{\alpha}_{\gamma\rho} T^{\rho}_{\cdot\beta} - G^{\rho}_{\gamma\beta} T^{\alpha}_{\cdot\rho}$$

is calculated using the asymmetric coefficients of connectivity $G^{\gamma}_{\alpha\beta}$; the coefficients of the second quadratic form of the surface $b_{\alpha\beta}$ are also, generally speaking, asymmetric.

Let us define the tensor $D^\tau \mathbf{T}_\Sigma$ in terms of its components

$$(D^\tau\mathbf{T}_\Sigma)^{\alpha}_{\cdot\beta} = \nabla^\tau T^{\alpha}_{\cdot\beta}\, d\tau, \quad (D^\tau\mathbf{T}_\Sigma)^{\alpha(n)} = T^{\alpha}_{\cdot\beta} h^{\beta(n)}\, d\tau$$

$$(D^\tau\mathbf{T}_\Sigma)_{(n)\beta} = T^{\alpha}_{\cdot\beta} h_{\alpha(n)}\, d\tau$$

where

$$\nabla^\tau T^{\alpha}_{\cdot\beta} = \dot{T}^{\alpha}_{\cdot\beta} + h^{\alpha}_{\rho\cdot} T^{\rho}_{\cdot\beta} - h^{\rho}_{\beta\cdot} T^{\alpha}_{\cdot\rho}$$

and where the components $h_{\alpha\cdot}^{\beta}, h^{\alpha(n)}$ of the tensor $\mathbf{h}_\Sigma$ can be regarded as components of the time derivatives of the local tangential basis vectors and of the normal to the surface.

In ther case of a Riemannian surface imbedded in the Euclidean space, the components of the tensor $\mathbf{h}_\Sigma$ are expressed in terms of the velocity vector components /28/ thus:

$$h^{\beta}_{\alpha\cdot} = \nabla_\alpha v^\beta - b_{\alpha\cdot}^{\beta} v^{(n)}, \quad h_{\alpha(n)} = \nabla_\alpha v^{(n)} + b_{\alpha\beta} v^\beta$$

The following expressions hold for the components of the tensor $\mathbf{W}_\Sigma = D^\tau D\mathbf{T}_\Sigma - DD^\tau\mathbf{T}_\Sigma$:

$$W^{\alpha}_{\cdot\beta} = (P_{\gamma\rho\cdot}^{\alpha} T^{\rho}_{\cdot\beta} - P_{\gamma\beta\cdot}^{\rho} T^{\alpha}_{\cdot\rho})\, d\tau\, d\eta^\gamma \tag{6.1}$$

$$W^{\alpha(n)} = P_{\gamma\beta(n)} T^{\alpha\beta}\, d\tau\, d\eta^\gamma, \quad W_{(n)\beta} = P_{\gamma\alpha(n)} T^{\alpha}_{\cdot\beta}\, d\tau\, d\eta^\gamma$$

where the components of the tensor $\mathbf{P}_\Sigma$ are given by

$$P_{\alpha\beta\cdot}^{\gamma} = \dot{G}^{\gamma}_{\alpha\beta} - \nabla_\alpha h_{\beta\cdot}^{\gamma} + b_{\alpha\cdot}^{\gamma} h_{\beta(n)} - b_{\alpha\beta} h^{\gamma(n)} \tag{6.2}$$

$$P_{\alpha\beta(n)} = \dot{b}_{\alpha\beta} - \nabla_\alpha h_{\beta(n)} - b_{\alpha\rho} h_{\beta\cdot}^{\rho}.$$

Formulas (6.2) and an expression analogous to (5.2) together yield the evolutionary equations for the components of the torsion and curvature tensor

$$\dot{S}_{\alpha\beta\cdot}^{\gamma} = P_{[\alpha\beta]\cdot}^{\gamma} + \nabla_{[\alpha} h_{\beta]\cdot}^{\gamma} - b_{[\alpha\cdot}^{\gamma} h_{\beta](n)} + b_{[\alpha\beta]} h^{\gamma(n)} \tag{6.3}$$

$$\dot{R}_{\alpha\beta\gamma\cdot}^{\delta} = 2\,[\nabla_{[\alpha} P_{\beta]\gamma\cdot}^{\delta} + \nabla_{[\alpha}\nabla_{\beta]} h_{\gamma\cdot}^{\delta} - \nabla_{[\alpha}(b_{\beta]\cdot}^{\delta} h_{\gamma(n)}) + \nabla_{[\alpha}(b_{\beta]\gamma} h^{\delta(n)}) +$$

$$S_{\alpha\beta\cdot}^{\rho}\,(P_{\rho\gamma\cdot}^{\delta} - \nabla_\rho h_{\gamma\cdot}^{\delta} - b_{\rho\cdot}^{\delta} h_{\gamma(n)} + b_{\rho\gamma} h^{\delta(n)})]$$

Let us find the components of the dislocation surface density $\boldsymbol{\alpha}_\Sigma$ and disclination surface density $\boldsymbol{\theta}_\Sigma$ tensors as follows:

$$\alpha^{(n)\gamma} = \varepsilon^{\alpha\beta} S_{\alpha\beta\cdot}^{\gamma}, \quad \alpha^{(n)(n)} = \varepsilon^{\alpha\beta} b_{\alpha\beta} \tag{6.4}$$

$$\theta^{(n)\gamma} = \varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\nabla_\alpha b_{\beta\delta} + \varepsilon^{\gamma\beta} b_{\alpha\beta}\alpha^{(n)\alpha}$$

$$\theta^{(n)(n)} = \tfrac{1}{2}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}(R_{\alpha\beta\gamma\delta} - b_{\alpha\delta} b_{\beta\gamma} + b_{\beta\delta} b_{\alpha\gamma})$$

and connect the quantities $\mathbf{h}_\Sigma$ and $\mathbf{P}_\Sigma$ with the dislocation $\mathbf{J}_\Sigma$ and disclination $\mathbf{S}_\Sigma$ surface flux tensors as follows:

$$J_{\alpha\cdot}^{\beta} = \nabla_\alpha v^\beta - b_{\alpha\cdot}^{\beta} v^{(n)} - h_{\alpha\cdot}^{\beta} \tag{6.5}$$

$$J_{\alpha(n)} = \nabla_\alpha v^{(n)} + b_{\alpha\beta} v^\beta - h_{\alpha(n)}$$

$$S_{\alpha\cdot}^{\beta} = -\varepsilon^{\beta\gamma} P_{\alpha\gamma(n)}, \quad S_{\alpha(n)} = -\tfrac{1}{2}\varepsilon^{\beta\gamma} P_{\alpha\beta\gamma}$$

Using the formulas (6.4), (6.5) we obtain, from (6.2), (6.3), the non-linear equations of the continuous theory of mobile defects in a two-dimensional medium

$$\begin{aligned}
\alpha^{\cdot(n)\gamma} &= -\varepsilon^{\alpha\beta}(\nabla_\alpha J_{\beta\cdot}^{\gamma} - b_{\alpha\cdot}^{\gamma} S_{\beta(n)}) + S^{\gamma(n)} - \alpha^{(n)\gamma} h_{\beta\cdot}^{\beta} - \\
&\quad \alpha^{(n)\beta}(\nabla_\beta v^\gamma - b_{\beta\cdot}^{\gamma} v^{(n)}) + \varepsilon^{\beta\gamma} v_{\beta} \theta^{(n)(n)} - \varepsilon^{\beta\gamma}\theta_{(n)\beta} v^{(n)} + \alpha^{(n)(n)} h^{\gamma(n)} \\
\alpha^{\cdot(n)(n)} &= -\varepsilon^{\alpha\beta}(\nabla_\alpha J_{\beta(n)} + b_{\alpha\gamma} J_{\beta\cdot}^{\gamma}) - S_{\varphi\cdot}^{\alpha} - \alpha^{(n)(n)} h_{\beta\cdot}^{\beta} - \\
&\quad \alpha^{(n)\beta}(\nabla_\beta v^{(n)} + b_{\beta\gamma} v^\gamma) + \varepsilon_{\beta\gamma}\theta^{(n)\beta} v^\gamma \\
\theta^{\cdot(n)\gamma} &= -\varepsilon^{\alpha\beta}(\nabla_\alpha S_{\beta\cdot}^{\gamma} - b_{\alpha\cdot}^{\gamma} S_{\beta(n)}) - \theta^{(n)\gamma} h_{\beta\cdot}^{\beta} - \theta^{(n)\beta} h_{\beta\cdot}^{\gamma} - \\
&\quad \alpha^{(n)\beta} S_{\beta\cdot}^{\gamma} + \theta^{(n)(n)} h^{\gamma(n)} \\
\theta^{\cdot(n)(n)} &= -\varepsilon^{\alpha\beta}(\nabla_\alpha S_{\beta(n)} + b_{\alpha\gamma} S_{\beta\cdot}^{\gamma}) - \alpha^{(n)\gamma} S_{\gamma(n)} - \theta^{(n)(n)} h_{\beta\cdot}^{\beta} - \theta^{(n)\alpha} h_{\alpha(n)}
\end{aligned} \tag{6.6}$$

When the non-linear terms are neglected the linear Eqs.(3.7) follow from (6.6).

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